Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 7 February 2008 (MN LaTeX style file v1.4)

On the r-mode spectrum of relativistic stars

Horst R. Beyer[1] and Kostas D. Kokkotas[2,3]
[1] *Universität Stuttgart, Institut A für Mechanik, Pfaffenwaldring 9, D-70550 Stuttgart, Germany.*
[2] *Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece.*
[3] *Max Planck Institute for Gravitational Physics, The Albert Einstein Institute, D-14473 Potsdam, Germany.*

Accepted 1999 (?). Received 1999 (?); in original form 1999

ABSTRACT
We present a mathematically rigorous proof that the r-mode spectrum of relativistic stars to the rotational lowest order has a continuous part. A rigorous definition of this spectrum is given in terms of the spectrum of a continuous linear operator. This study verifies earlier results by Kojima (1998) about the nature of the r-mode spectrum.

Key words: stellar oscillations, neutron stars, stellar instabilities

1 INTRODUCTION

The recently discovered r-mode instability (Andersson 1998a; Friedman & Morsink 1998) in rotating neutron stars, has significant implications on the rotational evolution of a newly-born neutron star. The r-modes are unstable due to the Chandrasekhar-Friedman-Schutz (CFS) mechanism (Chandrasekhar 1970; Friedman & Schutz 1978). Two independent computations by Andersson, Kokkotas & Schutz (1998) and Lindblom, Owen & Morsink (1998) find that the r-mode instability is responsible for slowing down a rapidly rotating, newly-born neutron star to rotation rates comparable to that of the initial period of the Crab pulsar (∼19 ms). This is achieved by the emission of current-quadrupole gravitational waves, which reduce the angular momentum of the star. The instability is active for as long as its growth-time is shorter than the damping-time due to the viscosity of neutron-star matter. The r-mode instability explains why only slowly-rotating pulsars are associated with supernova remnants. The r-mode instability does not allow millisecond pulsars to be formed after an accretion-induced collapse of a white dwarf (Andersson et al. 1998a). It seems that millisecond pulsars can only be formed by the accretion-induced spin-up of old, cold, neutron stars. Additionally, this instability should be active in accreting neutron stars and so can limit their rotation provided that the stars are hotter than about 2×10^5K and also in the rapidly spinning neutron stars in low mass X-ray binaries (LMXB)(Andersson et al. 1998b). Finally, while the initially rapidly rotating star spins down, an energy equivalent to roughly 1% of a solar mass is radiated in gravitational waves, making the process an interesting source of detectable gravitational waves (Owen et al. 1998).

Oscillations of stars are commonly described by the Lagrangian displacement vector $\vec{\xi}$, which describes the displacement of a given fluid element due to the oscillation. Since $\vec{\xi}$ is a vector on the (θ, ϕ) 2-sphere, it can be written as a sum of spheroidal and toroidal components (or polar and axial components, in a different terminology). In a non-rotating star, the usual f, p and g modes of oscillation are purely spheroidal, characterized by the indices (l, m) of the spherical harmonic function Y_l^m. In a rotating star, modes that reduce to purely spheroidal modes in the non-rotating limit, also acquire toroidal components. Conversely, r-modes in a non-rotating star are purely toroidal modes with vanishing frequency⋆.

In a rotating star, the displacement vector acquires spheroidal components and the frequency in the rotating frame, to first order in the rotational frequency Ω of the star, becomes

$$\sigma_r = \frac{2m\Omega}{l(l+1)} , \tag{1}$$

for a given (l, m) mode. An inertial observer, measures a frequency of

$$\sigma_i = m\Omega - \sigma_r . \tag{2}$$

As mentioned earlier, when the star is set in slow rotation the axial (toroidal) modes are no longer degenerate, but instead the new family of r-modes emerges, which are horizontal displacements on equipotential surfaces. In this case the axial (toroidal) perturbation with spherical harmonic indices (ℓ, m) induce polar (spheroidal) perturbations with harmonic indices $(\ell \pm 1, m)$ and vice versa.

The picture described above is purely Newtonian, the above calculation was performed using Newtonian slowly rotating stellar models and the power radiated in gravitational waves was estimated using the quadrapole formula. The two

⋆ In the relativistic case the picture is similar (Thorne & Campolattaro 1967) nevertheless, in this case there exist an additional family of quasinormal modes, the ones called "*spacetime or w-modes*"(Chandrasekhar & Ferrari 1991b; Kokkotas 1994).

approximations (Newtonian theory and slow rotation) that were used give only a qualitative picture while the quantitative results would change if at least general relativity is used. The assumption of slow rotation is a robust approximation because the expansion parameter $\epsilon = \Omega\sqrt{R^3/M}$ is usually very small and the fastest spinning known pulsar has $\epsilon \sim 0.3$.

The perturbation equations for slowly rotating relativistic stars were derived by Kojima (1992), see also (Chandrasekhar & Ferrari 1991a). Kojima (1993) calculated also the effect of slow rotation on f-modes. Andersson (1998) found the r-mode instability using the same set of equations, although his calculations overestimate the growth rate of the instability.

An important difference between Newtonian and general relativistic calculations is the dragging of the inertial frames, which might produce significant changes in the frequency spectra. This is exactly the case that we are studying in this article. To be more specific, Kojima (1998) suggested that if one calculates the r-mode frequencies using general relativity to lowest order in Ω the spectrum becomes continuous, in contrast to the calculations from Newtonian theory where the spectrum is discrete and the frequencies are given by the formula (1). *In this article we prove in a mathematically rigorous way that Kojima's suggestion for the existence of a continuous part within the spectrum is true.*

Continuous spectra have been found in many cases in the past in the study of differentially rotating fluids (Schutz & Verdaguer 1983; Verdaguer 1983; Balbinski 1984a; Balbinski 1984b). The continuous spectrum in these cases were again seen for the r-modes together with a wealth of interesting features such as: the passage of low-order r-modes from the discrete spectrum into the continuous one as the differential rotation increases; and the presence of low order discrete p-modes in the middle of the continuous spectrum in the more rapidly rotating disks (Schutz & Verdaguer 1983). *The stars under consideration here have no differential rotation and the existence of a continuous part of the spectrum is attributed to the dragging of the inertial frames due to general relativity.*

2 PERTURBATION EQUATIONS

Since our calculations will be based on the equations of Kojima (Kojima 1998; Kojima 1997) which are presented in detail there, here we are going only briefly to describe the perturbation equations.

We assume that the star is uniformly rotating with angular velocity $\Omega \sim O(\epsilon)$ where ϵ, as stated earlier, is small compared to unity. The metric is given by:

$$\begin{aligned} ds^2 = & - e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ & - 2\omega r^2 \sin^2\theta dt d\phi \ , \end{aligned} \tag{3}$$

where $\omega \sim O(\epsilon)$ describes the dragging of the inertial frame. If we include the effects of rotation only to order ϵ the configuration is still spherical, because the deformation is of order ϵ^2 (Hartle 1967). The star then is described by the standard Tolman-Oppenheimer-Volkov (TOV) equations (cf Chapter 23.5 (Misner et al. 1973)) plus an equation for ω

$$\left(jr^2\varpi'\right)' - 16\pi(\rho + p)e^{\lambda}jr^4\varpi = 0 \ , \tag{4}$$

where we have defined

$$\varpi = \Omega - \omega \tag{5}$$

a prime denotes derivative with respect to r, and

$$j = e^{-(\lambda+\nu)/2}. \tag{6}$$

In the vacuum outside the star ϖ can be written

$$\varpi = \Omega - \frac{2J}{r^3} \ , \tag{7}$$

where J is the angular momentum of the star. The function ϖ, both inside and outside the star is a function of r only and continuity of ϖ at the boundary (surface of the star, $r = R$) requires that $\varpi'_R = 6JR^{-4}$. Additionally, ϖ is monotonically increasing function of r limited to

$$\varpi_0 \le \varpi \le \Omega, \tag{8}$$

where ϖ_0 is the value at the center.

For the study of the perturbations of slowly rotating relativistic stars we should expand all perturbation functions in spherical harmonics and additionally assume a harmonic dependence of time $\exp[-i(\sigma t - m\phi)]$. Here, σ is, the normalized in units $(M/R^3)^{1/2}$, oscillation frequency. To lowest order for rotating stars, in accordance with the Newtonian theory, we expect that the toroidal perturbations of the fluid have finite frequencies of order Ω (or ϵ). Then from the six functions $h_0, h_1, H_0, H_1, H_2, K$ describing the metric perturbations in the Regge-Wheeler gauge (Thorne & Campolattaro 1967), only one, h_0, is of the same order as the function describing the toroidal fluid motions U †. All other perturbation functions plus the variations of the pressure δp and density $\delta\rho$ are of higher order and thus can be omitted. We should point out that this approximation is valid only for the study of the r-modes and is not appropriate for the study of other fluid or w-modes.

In a recent article Lockitch & Friedman (1998) suggest that rotation mixes in general all axial and polar perturbations and introduce the idea of *hybrid* modes. This mixing also eliminates all purely polar modes, but there is still a set of purely axial modes. They suggest that this set of purely axial modes should not exist for relativistic slowly rotating stars at least in the barotropic case. As we show here in the specific approximation described above it is possible that the r-mode spectrum has a continuous part.

Using the above assumptions the master equation governing quasi-toroidal oscillations is given by (Kojima 1998; Kojima 1997)

$$q\Phi + (\varpi - \mu)\left[v\Phi - \frac{1}{r^4 j}\left(r^4 j\Phi'\right)'\right] = 0 \ , \tag{9}$$

where

$$\Phi = \frac{h_0}{r^2} \ , \tag{10}$$

and

$$v = \frac{e^{\lambda}}{r^2}(l-1)(l+2) \ , \tag{11}$$

† The toroidal displacement vector is defined as

$$\vec{\xi} = \left(0, U_{\ell m}\sin^{-1}\theta\partial_\phi, -U_{\ell m}\partial_\theta\right) Y_\ell^m$$

$$q = \frac{1}{r^4 j}\left(r^4 j \varpi'\right)' = 16\pi(\rho + p)e^{\lambda}\varpi \ , \tag{12}$$

$$\mu = -\frac{l(l+1)}{2m}(\sigma - m\Omega). \tag{13}$$

Equation (9) in the Newtonian limit ($j \to 1$, $q \to 0$ and $\varpi \to \Omega$) reduces to a simple condition for the r-mode frequency $\varpi - \mu = 0$ which is identical with (2). To this order of Ω the eigenfrequency of the r-modes is independent of the radial dependence of the eigenfunction $\Phi(r)$ (or $h_0(r)$) and in this sense the eigenfrequency is infinitely degenerate.

The master equation (9) is not a regular eigenvalue problem since the coefficient $(\varpi - \mu)$ becomes singular inside the star for a certain value of μ. *The purpose of this work is to study the spectrum of this equation in a rigorous way.*

2.1 Conventions

The following **conventions** are used:
The symbols N, I, C denote the natural numbers (including zero), all real numbers greater than zero and the complex numbers, respectively. With r we will denote interchangeably some chosen element from I or else the identical mapping from I to the real numbers. The definition used will be clear from the context. For each $k \in N$ the symbol $C^k(I, C)$ denotes the linear space of k-times continuously differentiable complex-valued functions on I.

Throughout the paper Lebesgue integration theory in the formulation of (Riesz & Sz-Nagy 1955) is used. Compare with respect to this also Chapter III in (Hirzebruch & Scharlau 1971) and Appendix A in (Weidmann 1976). Following common usage there is no difference made between an almost everywhere (with respect to the chosen measure) defined function f and its associated equivalence class (consisting of all almost everywhere defined functions which differ from f only on a set of measure zero). In this sense $L^2_C\left(I, r^4 j\right)$ denotes the Hilbert space of complex-valued, square integrable functions (with respect to the measure $r^4 j\, dr$) on the real line. The scalar product $< \mid >$ on $L^2_C\left(I, r^4 j\right)$ is defined by

$$< f|g > := \int_0^\infty r^4 j\, f^* g dr \tag{14}$$

for all $f, g \in L^2_C\left(I, r^4 j\right)$. Finally, $L^2_C\left(I^2\right)$ denotes the Hilbert space of complex-valued, with respect to the Lebesgue measure square integrable functions on the two-dimensional interval I^2.

3 THE SPECTRUM OF KOJIMA'S MASTER EQUATION

In the following and in the remainder of the paper we consider the cases $l \geq 1$ only. Then, in a first step, we rewrite (9) by introducing a new dependent variable φ defined by:

$$\varphi := v\Phi - \frac{1}{r^4 j}\left(r^4 j \Phi'\right)' \ . \tag{15}$$

The corresponding representation of Φ in terms of φ can be performed with the help of special linear independent solutions Φ_1, Φ_2 of the differential equation

$$\frac{1}{r^4 j}\left(r^4 j \Phi'\right)' - v\Phi =$$

$$\Phi'' + \left(\frac{4}{r} + \frac{j'}{j}\right)\Phi' - \frac{(l-1)(l+2)}{r^2}e^{\lambda}\Phi = 0 \ , \tag{16}$$

given in the next section. This representation is given by:

$$\begin{aligned}\Phi(r) \quad := \quad & -\frac{1}{W}\left[\Phi_2(r)\int_0^r \Phi_1 r'^4 j\varphi dr' \right. \\ & \left. + \Phi_1(r)\int_r^\infty \Phi_2 r'^4 j\varphi dr'\right] \ ,\end{aligned} \tag{17}$$

where W is the Wronskian of Φ_1 and Φ_2, defined by

$$W := r^4 j\left(\Phi_1 \Phi_2' - \Phi_1' \Phi_2\right) \ . \tag{18}$$

Roughly, Φ_1 is an element of $L^2_C\left(I, r^4 j\right)$ for small r and Φ_2 is an element of $L^2_C\left(I, r^4 j\right)$ for large r. Apart from an irrelevant factor such functions are uniquely defined by these demands. The use of these functions in the inversion (17) of (15) is necessary because of the demand (boundary condition) that Φ is an element of $L^2_C\left(I, r^4 j\right)$.

Introducing the new variable φ into (9) leads to the following equation for φ:

$$\begin{aligned}& -\frac{q}{W}\left[\Phi_2(r)\int_0^r \Phi_1 r'^4 j\varphi dr' + \Phi_1(r)\int_r^\infty \Phi_2 r'^4 j\varphi dr'\right] \\ & +(\varpi(r) - \mu)\varphi(r) = 0 \, ,\end{aligned} \tag{19}$$

for all $r \in I$.

In a second step (19) is turned into a regular spectral problem for the continuous linear operator A on $L^2_C\left(I, r^4 j\right)$ defined as follows:
For every $f \in L^2_C\left(I, r^4 j\right)$ we define $Af \in L^2_C\left(I, r^4 j\right)$ by

$$\begin{aligned}(Af)(r) := \quad & - \quad \frac{q}{W}\left[\Phi_2(r)\int_0^r \Phi_1 r'^4 j f dr' \right. \\ & + \quad \left. \Phi_1(r)\int_r^\infty \Phi_2 r'^4 j f dr'\right] + \varpi(r) f(r) \, ,\end{aligned} \tag{20}$$

for all $r \in I$. That this indeed defines a continuous linear operator on the whole of $L^2_C\left(I, r^4 j\right)$ can be seen as follows. First, obviously, since ϖ is bounded continuous, by

$$T_\varpi f := \varpi f \, , f \in L^2_C\left(I, r^4 j\right) \tag{21}$$

there is defined a continuous linear operator on $L^2_C\left(I, r^4 j\right)$. Second, in the Appendix A it will be shown that from

$$\begin{aligned}(Bf)(r) := \quad & - \quad \frac{q}{W}\left[\Phi_2(r)\int_0^r \Phi_1 r'^4 j f dr' \right. \\ & + \quad \left. \Phi_1(r)\int_r^\infty \Phi_2 r'^4 j f dr'\right] \, ,\end{aligned} \tag{22}$$

for all $r \in I$ and every $f \in L^2_C\left(I, r^4 j\right)$ there is even defined a Hilbert-Schmidt operator B on $L^2_C\left(I, r^4 j\right)$. Hence B is not only continuous but in addition *compact* and Hilbert-Schmidt. As a consequence (20) defines a continuous linear operator on A on $L^2_C\left(I, r^4 j\right)$, being equal to the sum of T_ϖ and B.

The determination of the spectrum of (9) is now reduced to finding the spectrum $\sigma(A)$ of the continuous linear (non-self-adjoint) operator A. Since A is continuous it follows that $\sigma(A)$ is a *bounded* subset of the complex plane contained in a circle around the origin with the radius given by the operator norm $\|A\|$ of A (Reed & Simon 1978). The so called essential spectrum of the operator T_ϖ is given by the values

$$\varpi_0 \leq \mu = -\frac{\ell(\ell+1)}{2m}\,(\sigma - m\Omega) \leq \Omega\,. \tag{23}$$

(see (Reed & Simon 1978; Reed & Simon 1980)) Moreover it is known (see e.g. (Reed & Simon 1978) Corollary 2 of Theorem XIII.14 in Vol. IV) that the essential spectrum ‡ is invariant under perturbations by a compact linear operator such as B. Hence the essential spectrum $\sigma_{ess}(A)$ of A, which is a part of $\sigma(A)$, is also given by the range of values (23). The complement $\sigma_{disc}(A) := \sigma(A) \setminus \sigma_{ess}(A)$ (which is possibly empty) consists of *isolated* eigenvalues of finite multiplicity (Reed & Simon 1978). Using an argument of Kojima (1998) it follows that these eigenvalues are real (hence $\sigma(A)$ is also real) and are contained in the interval $(\Omega, \|A\|]$.

4 DETERMINATION OF Φ_1 AND Φ_2

Here we distinguish two cases.

The first uses the following assumptions on ρ and p : Both, p and ρ are continuous real-valued functions on I satisfying the Tolman-Oppenheimer-Volkov equations and in addition are such that the limits

$$\lim_{r\to 0}\rho(r) \quad \text{and} \quad \lim_{r\to 0}p(r) \tag{24}$$

both exist and

$$\rho(r) = 0 \quad \text{and} \quad p(r) = 0 \quad \text{for} \quad r \geq R\,, \tag{25}$$

are both satisfied, where R denotes the radius of the star. § Under these conditions the existence of the special solutions Φ_1, Φ_2 of the differential equation (16) is given by a theorem of Dunkel (Dunkel 1912) (compare also (Levinson 1948), (Bellman 1949) and (Hille 1969)) on linear first-order systems of differential equations having asymptotic constant coefficients at $+\infty$. By transformation one gets from this a theorem where the singular point is finite. This theorem - which for the reader's convenience is given in the Appendix A - generalizes well - known results on weakly singular linear first-order systems with analytic coefficients. The determination of Φ_1 and Φ_2 proceeds now as follows. First from the TOV equations it follows that λ, j are continuously differentiable functions on I such that both:

$$\frac{j'}{j} \quad \text{and} \quad \frac{e^{\lambda}-1}{r} \tag{26}$$

are continuous as well as *Lebesgue integrable near* 0 and that both

$$r^2\,\frac{j'}{j} \quad \text{and} \quad r\left(e^{\lambda}-1\right) \tag{27}$$

are continuous as well as *Lebesgue integrable near* ∞. As a consequence of the theorem given in the Appendix A follows the existence of linearly independent solutions Φ_1, Φ_3 of (16) satisfying

$$\begin{aligned}
\lim_{r\to 0} r^{1-l}\Phi_1(r) &= 1 \\
\lim_{r\to 0} r^{2-l}\Phi_1'(r) &= l-1 \\
\lim_{r\to 0} r^{l+2}\Phi_3(r) &= 1 \\
\lim_{r\to 0} r^{l+3}\Phi_3'(r) &= -(l+2)
\end{aligned} \tag{28}$$

and of linearly independent solutions Φ_2, Φ_4 of (16) satisfying

$$\begin{aligned}
\lim_{r\to\infty} r^{l+2}\Phi_2(r) &= 1 \\
\lim_{r\to\infty} r^{l+3}\Phi_2'(r) &= -(l+2) \\
\lim_{r\to\infty} r^{1-l}\Phi_4(r) &= 1 \\
\lim_{r\to\infty} r^{2-l}\Phi_4'(r) &= l-1\,.
\end{aligned} \tag{29}$$

In the next step we conclude that Φ_1, Φ_2 are linear independent. The remaining solutions Φ_3, Φ_4 will be important in the proof of the compactness of B (see appendix).

Up to now everything said in this section is also valid for the case $l = 0$. For the following proof of linear independence we have to assume that $l \geq 1$. This is assumed in the remainder of the paper.

The proof of the linear independence of Φ_1, Φ_2 proceeds indirectly. So we assume on the contrary that there is a non-vanishing real α such that $\Phi_1 = \alpha\Phi_2$. Using this along with (28), (29), Lebesgue's dominated convergence theorem and the monotonous convergence theorem we conclude:

$$\begin{aligned}
0 &= \int_0^\infty \Phi_1\left[-\left(r^4 j\Phi_1'\right)' + (l-1)(l+2)r^2 j e^{\lambda}\Phi_1\right]dr \\
&= \lim_{n\to\infty}\left[-r^4 j\Phi_1\Phi_1'\,\Big|_{R/n}^{nR} + \int_{R/n}^{nR} r^4 j\Phi_1'^2 dr\right] \\
&+ (l-1)(l+2)\int_0^\infty r^2 j e^{\lambda}\Phi_1^2 dr \\
&= \int_0^\infty r^4 j\Phi_1'^2 dr + (l-1)(l+2)\int_0^\infty r^2 j e^{\lambda}\Phi_1^2 dr\,.
\end{aligned}$$

Where we have made use of the fact that the function

$$r^2 j e^{\lambda}\Phi_1^2 \tag{30}$$

is Lebesgue integrable on I. This can be concluded from the facts that the function j has a continuous extension onto the closed interval $[0, \infty)$, that the functions λ and j are constant for $r \geq R$ and that

$$\begin{aligned}
\lim_{r\to 0} e^{\lambda(r)} &= 1 \\
\lim_{r\to\infty} e^{\lambda(r)} &= 1\,.
\end{aligned} \tag{31}$$

All these facts are consequences of the TOV equations and the assumptions made on p and ρ. Hence it follows from (30) the **contradiction** that the function Φ_1 is trivial. As a consequence the functions Φ_1, Φ_2 are linear independent.

The second case considers slowly rotating homogeneous stellar models. In this case the functions e^{λ} and j can be given explicitly in terms of elementary functions (see e.g., (Chandrasekhar & Miller 1974)). It turns out that λ, j are continuous functions and that j is continuously differentiable on $I \setminus \{R\}$ such that the limits at R of the derivative of j from the right and from the left, respectively, differ from each other. The functions (26) are also *Lebesgue integrable near* 0 and (27) are also *Lebesgue integrable near* ∞. In addition (31) is also valid for this case. As a consequence of the theorem in the Appendix A follows the existence of continuously

‡ Defined according to (Reed & Simon 1978) on p. 106 of Vol. IV .

§ These assumptions are satisfied for instance for slowly rotating polytropic stellar models (see e.g., (Hartle 1967)).

differentiable functions Φ_1, Φ_2 on I which are two times continuously differentiable on $I \setminus \{R\}$, satisfy (16) on $I \setminus \{R\}$ and in addition satisfy (28), (29). The linear independence of these functions can be proved completely analogous to the previously considered case.

5 DISCUSSION

In the previous section we showed for a wide class of background models for slowly rotating stars that for each $r_0 \in I$ the corresponding $\mu = \varpi(r_0)$ (or rather the corresponding σ according to (13)) belongs to the spectrum of (9). Furthermore we achieved a *precise definition for the spectrum of* (9) *as the spectrum of the linear operator A*.

This is an interesting new result that needs further study. There are still questions to be answered in order to understand the effect of the frame dragging induced by general relativity to the spectrum of the rotating stars. For example, one cannot exclude the possibility that isolated eigenvalues might also exist (Andersson 1998b). Also, the specific form of equation (9) is found under the assumption that the r-mode frequencies are of order Ω. This implies that a few terms involving toroidal motions and all the terms related to spheroidal motions have been omitted because they were either either of higher order or because they contribute via higher harmonics $(l \pm 1, m)$. This is not the case with the hybrid modes found by Lockitch & Friedman (1998) where the spheroidal motions have been taken into account. If one includes these extra terms the form of the equation will change and the effect on the spectrum will be the emergence of the hybrid modes, but still the underlying nature of the spectrum can be the same and the techniques applied here can be used as well.

Moreover, in the more general case, there is an imaginary part for each frequency which corresponds to the damping or growth of the fluid motions due to the emission of gravitational radiation or due to the viscosity, so it will be interesting to examine whether the spectrum will still be continuous.

In other words the discovery of the existence of a continuous part of the r-mode spectrum is just the first step towards understanding the real nature of the spectrum. Already the idea of hybrid modes (Lockitch & Friedman1998; Lindblom & Ipser 1998) adds new features in the spectrum that have been overlooked in all previous studies. More work towards identifying possible isolated eigenvalues of the spectrum should be done and additionally there is need to examine if this specific nature of the spectrum is preserved when terms of order Ω^2 will be included in equation (9), the coexistence of continuous and discrete part of the spectrum is not at all impossible.

Concluding, we would like to point out that the existence of a continuous part of the spectrum might affect some of the astrophysical estimations being made for the growth time of the r-mode instability and consequently all the Newtonian estimations being made earlier by (Andersson et al. 1998a; Kokkotas & Stergioulas 1998; Andersson et al. 1998b).

ACKNOWLEDGMENTS

We would like to thank N. Andersson, J. L. Friedman, K.H. Lockitch, B. G. Schmidt, B. F. Schutz, and N. Stergioulas for helpful discussions, during our visit to Albert Einstein Institute, Potsdam. We are grateful to G. Allen for the critical reading of the manuscript.

REFERENCES

Andersson N., 1998a, ApJ, 502, 708
Andersson N., 1998b, private communication
Andersson N., Kokkotas K. D. & Schutz B. F., 1998, Ap.J., in press, astro-ph/9805225
Andersson N., Kokkotas K. D. & Stergioulas N., 1998, ApJ, in press, astro-ph/9806089
Balbinski E., 1984, M.N.R.A.S., 209, 145
Balbinski E., 1984, M.N.R.A.S., 209, 721
Bellman R., 1949, *A survey of the theory of the boundedness, stability, and asymptotic behaviour of solutions of linear and nonlinear differential and difference equations* (Washington DC: NAVEXOS P-596, Office of Naval Research)
Chandrasekhar S., 1970, Phys. Rev. Lett., 24, 61
Chandrasekhar S. & Ferrari V., 1991a, Proc. R. Soc. Lond., A433, 423
Chandrasekhar S. & Ferrari V., 1991b Proc. R. Soc. Lond., A434, 449
Chandrasekhar S. & Miller J. C., 1974, M.N.R.A.S., 167, 63
Dunkel O., 1912, Am. Acad. Arts Sci. Proc., 38, 341
Friedman J. L. & Schutz B. F., 1978, Ap.J., 22, 281
Friedman J. L. & Morsink S., 1998, ApJ, 502, 7145
Hartle J. B., 1967, ApJ, 150, 1005
Hille E., 1969, *Lectures on ordinary differential equations* (Reading: Addison-Wesley)
Hirzebruch F. & Scharlau W., 1971, *Einführung in die Funktionalanalysis* (Mannheim: BI)
Kojima Y., 1992, Phys. Rev. D., 46, 4289
Kojima Y., 1993, Ap.J., 414, 247
Kojima Y., 1997, Prog. Theor. Physics Suppl., 128, 251
Kojima Y., 1998, M.N.R.A.S., 293, 49
Kokkotas K.D., 1994, M.N.R.A.S., 268, 1015, Erratum: 1995, 277, 1599
Kokkotas K.D. & Stergioulas N., 1998, A&A, 341, 110
Levinson N., 1948, Duke Math. J., 15, 111
Lindblom L. & Ipser J.R., 1998, preprint, gr-qc/9807049
Lindblom L., Owen B. J. & Morsink S., 1998, Phys. Rev. Lett., 80, 4843
Lockitch K.H. & Friedman J.L., 1998, preprint, astro-ph/9812019
Misner C. W., Thorne K. S. & Wheeler J. A., 1973, *Gravitation* W.H.Freeman
Owen B., Lindblom L., Cutler C., Schutz B.F., Vecchio A. & Andersson N., 1998, Phys. Rev. D, in press, gr-qc/9804044
Schutz B. F. & Verdaguer E., 1983, M.N.R.A.S., 202, 881
Reed M. & Simon B., 1978 *Methods of Modern Mathematical Physics Volume IV* (New York: Academic)
Reed M. & Simon B., 1980 *Methods of Modern Mathematical Physics Volume I* (New York: Academic)
Riesz F. & Sz-Nagy B., 1955, *Functional Analysis* (New York: Unger)
Thorne K. S.& Campolattaro A., 1967, ApJ., 149, 591
Verdaguer E., 1983, M.N.R.A.S., 202, 903
Weidmann J., 1976, *Lineare Operatoren in Hilberträumen* (Teubner: Stuttgart)

APPENDIX A: PROOF OF THE THEOREM

The variant of the theorem of Dunkel (Dunkel 1912) (compare also (Levinson 1948; Bellman 1949; Hille 1969)) used in Section 4 is the following.

Theorem: Let $n \in N$; $a, t_0 \in R$ with $a < t_0$; $\mu \in N$; $\alpha_\mu := 1$ for $\mu = 0$ and $\alpha_\mu := \mu$ for $\mu \neq 0$. In addition let A_0 be a diagonalizable complex $n \times n$ matrix and $e'_1, \ldots, e'_n$ be a basis of C^n consisting of eigenvectors of A_0. Further, for each $j \in \{1, \cdots, n\}$ let λ_j be the eigenvalue corresponding to e'_j and P_j be the matrix representing the projection of C^n onto $C.e'_j$ with respect to the canonical basis of C^n. Finally, let A_1 be a continuous map from (a, t_0) into the complex $n \times n$ matrices $M(n \times n, C)$ for which there is a number $c \in (a, t_0)$ such that the restriction of A_{1jk} to $[c, t_0)$ is Lebesgue integrable for each $j, k \in 1, ..., n$.

Then there is a C^1 map $R : (a, t_0) \to M(n \times n, C)$ with $lim_{t \to 0} R_{jk}(t) = 0$ for each $j, k \in 1, \ldots, n$ and such that $u : (a, t_0) \to M(n \times n, C)$ defined by

$$u(t) := \begin{cases} \sum_{j=1}^{n} (t_0 - t)^{-\lambda_j} \cdot (E + R(t)) \cdot P_j \text{ for } \mu = 0 \\ \sum_{j=1}^{n} exp(\lambda_j (t_0 - t)^{-\mu}) \cdot (E + R(t)) \cdot P_j \text{ for } \mu \neq 0 \end{cases} \tag{A1}$$

for all $t \in (a, t_0)$ (where E is the $n \times n$ unit matrix), maps into the invertible $n \times n$ matrices and satisfies

$$u'(t) = \left(\frac{\alpha_\mu}{(t_0 - t)^{\mu+1}} A_0 + A_1(t) \right) \cdot u(t) \tag{A2}$$

for each $t \in (a, t_0)$.

Now we show that by (22) for all $r \in I$ and every $f \in L^2_C\left(I, r^4 j\right)$ there is defined a Hilbert-Schmidt operator B on $L^2_C\left(I, r^4 j\right)$. Using the unitary transformation U from $L^2_C\left(I, r^4 j\right)$ to $L^2_C(I)$ given by

$$Uf := r^2 \sqrt{j} f \ , \qquad f \in L^2_C\left(I, r^4 j\right) \tag{A3}$$

it easily seen that this is equivalent to showing that the integral operator $Int(K')$ with the kernel function K', where

$$K'(r, r') := -\frac{q(r)(rr')^2 (j(r)j(r'))^{1/2}}{W} \begin{cases} \Phi_2(r)\Phi_1(r') \text{ for } r' \leq r \\ \Phi_1(r)\Phi_2(r') \text{ for } r' > r. \end{cases} \tag{A4}$$

for all $r \in I$ and $r' \in I$, defines a Hilbert-Schmidt operator on $L^2_C(I)$. It is well-known (see e.g. (Reed & Simon 1978; Reed & Simon 1980)), that this is equivalent to showing that K' is an element of $L^2_C\left(I^2\right)$. Since K' is continuous this follows if we can show that $|K'|^2$ is integrable over I^2. For this we notice that as a consequence of (28), (29), there are positive real c_1, c_2, such that

$$\begin{aligned} |\Phi_1| &\leq c_1 r^{l-1} \\ |\Phi_2| &\leq c_2 r^{-(l+2)} . \end{aligned} \tag{A5}$$

From this and since j is (by the TOV equations) bounded, the integrability of $|K'|^2$ follows as an application of Lebesgue's dominated convergence theorem if the integrability of the following auxiliary function H can be shown:

$$H(r, r') := |q(r)|^2 \begin{cases} r^{-2l} (r')^{2(l+1)} \text{ for } r' \leq r \\ r^{2(l+1)} (r')^{-2l} \text{ for } r' > r. \end{cases} \tag{A6}$$

Now for each $r \in I$:

$$\int_0^\infty H(r, r') dr' = \frac{2(2l+1)}{(2l-1)(2l+3)} r^3 |q(r)|^2 \tag{A7}$$

and this expression is integrable over I since q is continuous and has a compact support. Hence the integrability of H follows from Tonelli's Theorem. Collecting everything, we conclude that B defines a Hilbert-Schmidt operator on $L^2_C\left(I, r^4 j\right)$.